

Calculating optimal limits for transacting credit card customers

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Abstract

We present a model of credit card profitability, assuming that the card-holder always pays the full outstanding balance. The motivation for the model is to calculate an optimal credit limit, which requires an expression for the expected outstanding balance. We derive its Laplace transform, assuming that purchases are made according to a marked point process and that there is a simplified balance control policy in place to prevent the credit limit being exceeded. We calculate optimal limits for a compound Poisson process example and show that the optimal limit scales with the distribution of the purchasing process and that the probability of exceeding the optimal limit remains constant. We establish a connection with the classic newsvendor model and use this to calculate bounds on the optimal limit for a more complicated balance control policy. Finally, we apply our model to real credit card purchase data.

Keywords: Banking; finance; optimization; stochastic processes; statistics; time series.

Introduction

Managers of retail credit card portfolios regularly employ modelling techniques to aid and automate decision-making throughout the customer life-cycle. At the point where the customer is acquired, the decision is whether or not to grant credit, and, if credit is to be granted, what amount. Later in the customer life-cycle, incentives such as an increased limit or a different interest rate may be offered, so the problem is to determine those customers most likely to accept the offer, or those who will generate the most profit if they accept.

There are numerous techniques available for modelling such decisions. For a review of the most common techniques in use, see Rosenberg and Gleit (1994), Thomas et al. (2002, 2004), Crook et al. (2007), or Hand and Henley (1997) for a particular emphasis on statistical techniques. The particular problem of credit limit assignment has been analysed by several authors.

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Bierman and Hausman (1970) formulated a dynamic programming model in which the decision variables were whether or not to grant credit and what amount. In their formulation, the amount of credit offered was linked to the probability of non-payment via an exponentially declining relationship. Their model was later extended in Dirickx and Wakeman (1976) to remove the assumption that there is a zero expected future payoff from period i onwards if no payment is made. More recently, Trench et al. (2003) developed a Markov decision process (MDP) in which the objective was to optimise customer lifetime value by either changing the customer's credit limit or interest rate. So and Thomas (2011) also used a MDP to generate a dynamic credit limit policy where the state space was the account behaviour score.

The problem that we address in this paper differs from those mentioned above in that we are interested in the situation where credit has already been granted and a limit assigned. A credit risk manager or analyst may want to review a customer's limit if, for example, the customer has contacted the card issuer to request an increased limit or request a review of an automated decision to decline a limit increase. We suppose that individual transaction data (the history of purchases and payments) has been collected and retained and the question is therefore whether we should revise an individual customer's limit in light of their spending and payment behaviour with the objective of maximising profitability for the bank. In particular, we focus on the situation where the customer exhibits *transacting* behaviour. That is, the customer regularly pays the outstanding balance in full by the due date issued on their monthly statement.

We propose modelling at the individual level using transaction data as a way to create further differentiation within certain populations encountered in credit card portfolios. It is quite common to see large numbers of low-risk accounts with similar behaviour scores and since this is often a key input into account management strategies, these accounts will receive identical treatment. Analysis of the individual transaction patterns of such accounts can provide increased differentiation and lead to the development of more profitable strategies.

Indeed, the availability of transaction-level data has soared in the last decade with the increasing prevalence of data warehouses in financial institutions. Despite this increased availability, there have been few attempts made to utilise this data to develop new models for account management. Hand and Blunt (2001) detailed the use of data mining techniques to reveal spending patterns in a database of UK credit card transactions. In particular, they modelled spending behaviours at petrol stations. Till and Hand (2003) also used a database of credit card transactions at petrol stations and fitted various distributions to the inter-transaction times. This was explored in more detail in Till (2001).

The model developed in this paper bears similarities to those used in

inventory theory, in particular, the model analysed in Arrow et al. (1951), which is more commonly referred to as the newsvendor model and originally attributed to Edgeworth (1888). The newsvendor model is known to have a solution in terms of the quantile function of the input demand distribution, but the differences introduced by our model formulation require a solution by other means. We follow a method similar to that used in Chiera and Taylor (2002) to obtain a solution in terms of Laplace transforms, which we then invert numerically following the algorithm described in Abate and Whitt (1995).

A model of individual credit card profitability

Consider a credit card with limit $\ell > 0$ and no annual fees or loyalty scheme. Let $B_\ell(t)$ denote the outstanding balance at time t and let $R(t)$ denote the cumulative profit earned up to time t . We assume that the lending institution must pay a proportion of the limit ν as a cost of financing. Now, let $0 = t_0 < t_1 < \dots < t_n$ be a sequence of billing times where the outstanding balance is billed to the customer and s_1, \dots, s_n be a sequence of times by which full or partial payment of the outstanding balance is due, with each $s_i \in (t_i, t_{i+1})$, $i = 1, \dots, n$. The times s_i are commonly referred to as due dates, the interval (t_i, t_{i+1}) as the i th statement period, and the interval (t_i, s_i) as the i th interest-free period. To simplify matters, we assume that $s_i = t_i$, $i \geq 1$, but note this does not affect the generality of our model as we shall see.

If we further assume that the customer exhibits transacting behaviour and pays the full balance due before the end of each statement period, then there will be no interest charges and the only contribution to revenue will be from interchange which we assume occurs at a proportion γ of the total purchase amount. The only cost will be the cost of funding the limit assigned to the customer and so the profit earned in the period (t_i, t_{i+1}) is

$$R(t_{i+1}) - R(t_i) = \gamma B_\ell(t_{i+1}) - \nu\ell, \quad 0 \leq i \leq n, \quad (1)$$

since the only contribution to the balance will be from new purchases in the period (t_i, t_{i+1}) .

Assume now that the each statement period has fixed length $T > 0$ and that the customer's purchasing behaviour remains the same across each period. Then we need only consider a single period and we may rewrite (1) as

$$R(T) = \gamma B_\ell(T) - \nu\ell. \quad (2)$$

Now taking the expectation of (2), we have

$$\mathbf{E}[R(T)] = \gamma \mathbf{E}[B_\ell(T)] - \nu\ell. \quad (3)$$

If there is an interest-free period of fixed length $b > 0$, then it is not hard to see that we may account for its effect by adding this to the length of the statement period.

We now seek to maximise the expected revenue by finding an appropriate limit $\ell \in \Lambda$ where Λ is a set of permitted limits. Define

$$\hat{\ell} := \arg \max_{\ell \in \Lambda} \left\{ \gamma \mathbf{E}[B_\ell(T)] - \nu \ell \right\}. \quad (4)$$

If Λ is a finite set, then we may determine $\hat{\ell}$ by simply evaluating the right-hand side of (4) at each point in Λ . The situation where Λ is not countable requires some knowledge of the properties of $\mathbf{E}[B_\ell(T)]$. If $\mathbf{E}[B_\ell(T)]$ happens to be a differentiable function of ℓ at the point T , and if the maximum in (4) occurs in the interior of Λ , we can determine $\hat{\ell}$ by differentiating (3) and setting the right-hand side equal to 0. This yields

$$\frac{\nu}{\gamma} = \frac{\partial}{\partial \ell} \mathbf{E}[B_\ell(T)]. \quad (5)$$

The task is now to find the limit $\hat{\ell}$ that will render the right-hand side of (5) equal to the left-hand side. Whether we are solving (4) in the general case, or determining $\hat{\ell}$ via (5), we require an expression for $\mathbf{E}[B_\ell(T)]$ or its derivative if it exists. We develop such expressions in the next section.

An integral equation for the tail function

We assume that the card-holder attempts to make purchases according to a marked point process

$$A(t) = \sum_{i=1}^{N(t)} \xi_i, \quad (6)$$

where $\{\xi_i\}_{i=1,2,\dots}$ is a sequence of non-negative, independent random variables with common distribution function F and $N(t)$ is a random variable, independent of $\{\xi_i\}_{i=1,2,\dots}$, describing the number of events in $(0, t]$ in a renewal process with inter-event time distribution G . For $k = 1, 2, \dots$, we write $t_k = \inf\{t : N(t) = k\}$ and $\tau_k = t_k - t_{k-1}$, with $t_0 = 0$. For the remainder of the paper, we assume that both F and G are of exponential order and that all moments of the distributions exist. These conditions are sufficient to ensure the existence of the Laplace transforms

$$\tilde{f}(\theta) = \int_0^\infty e^{-\theta z} F(dz) \quad \text{and} \quad \tilde{g}(\omega) = \int_0^\infty e^{-\omega u} G(du) \quad (7)$$

for $\operatorname{Re}(\theta) > \sigma_F$ and $\operatorname{Re}(\omega) > \sigma_G$, where the respective abscissae of convergence σ_F and σ_G of \tilde{f} and \tilde{g} are strictly less than zero.

Suppose now that the bank enforces a control policy on the outstanding balance whereby if an attempted purchase would cause the outstanding balance to exceed the credit limit ℓ , that purchase is rejected and the customer is barred from making any further purchases until the outstanding balance is repaid at the end of the statement period. Since our customer is a transactor, payment of the outstanding balance in full is guaranteed.

We find an expression for the tail function $S_\ell(y, t) := \mathbf{P}(B_\ell(t) \in (y, \ell])$ by conditioning on the time and value of the first jump of the process. For $0 < y \leq \ell$, we have three possibilities to consider

- i. The process jumps to some $z \in (0, y]$ at some time $u \in (0, t)$ and then regenerates itself at this point. That is, a new process starts at z that behaves like the original one, but shifted by $\ell - z$ in space and $t - u$ in time.
- ii. The process jumps to some $z \in (y, \ell]$ and any subsequent jumps that happen in the remaining time interval (u, t) cannot take the process out of the interval $(y, \ell]$.
- iii. The process jumps to some $z \in (\ell, \infty)$. If this occurs, the process is frozen at the point from which it jumped.

With $\tau = \tau_1$ and $\xi = \xi_1$ we combine the cases above to derive

$$\mathbf{E} [\mathbf{1}_{\{B_\ell(t) \in (y, \ell]\}} \mid \tau, \xi] = \begin{cases} \mathbf{1}_{\{B_\ell(t) - B_\ell(\tau) \in (y - \xi, \ell - \xi]\}}, & \tau \leq t, \xi \leq y \\ 1, & \tau \leq t, y < \xi \leq \ell \end{cases} \quad (8)$$

By the regenerative property mentioned above, the distribution of $B_\ell(t) - B_\ell(\tau)$ is the same as that of $B_{\ell-\xi}(t - \tau)$, which assumes that the payment period is $t - \tau$ and the credit limit is $\ell - \xi$. So we have

$$\mathbf{E} [\mathbf{1}_{\{B_\ell(t) \in (y, \ell]\}} \mid \tau, \xi] = \begin{cases} \mathbf{1}_{\{B_{\ell-\xi}(t-\tau) \in (y-\xi, \ell-\xi]\}}, & \tau \leq t, \xi \leq y \\ 1, & \tau \leq t, y < \xi \leq \ell \end{cases} \quad (9)$$

By the law of total probability,

$$S_\ell(y, t) = \int_0^t \int_0^y S_{\ell-z}(y-z, t-u) F(dz) G(du) + G(t)(F(\ell) - F(y)). \quad (10)$$

We now wish to obtain the Laplace transform

$$\tilde{S}(\theta, \omega, \psi) := \int_0^\infty \int_0^\infty \int_0^\ell e^{-(\omega t + \theta \ell + \psi y)} S_\ell(y, t) dy d\ell dt. \quad (11)$$

It follows from (10) that

$$|S_\ell(y, t)| \leq G(t)F(y) + G(t)F(\ell) - G(t)F(y) = G(t)F(\ell), \quad (12)$$

and since products of functions of exponential order are also of exponential order, we have that $S_\ell(y, t)$ is of exponential order when F and G are, and hence, the Laplace transform (11) exists.

Applying the Laplace transform to (10), we have after some rearrangement,

$$\tilde{S}(\theta, \omega, \psi) = \frac{1}{\theta\omega\psi} \frac{\tilde{g}(\omega)(\tilde{f}(\theta) - \tilde{f}(\theta + \psi))}{1 - \tilde{g}(\omega)\tilde{f}(\theta + \psi)}. \quad (13)$$

To calculate the (two-dimensional) Laplace transform of the expectation, we note that

$$\mathcal{L}_{\theta, \omega}\{\mathbf{E}[B_\ell(t)]\} := \int_0^\infty \int_0^\infty e^{-(\omega t + \theta\ell)} \mathbf{E}[B_\ell(t)] d\ell dt \quad (14)$$

$$= \int_0^\infty \int_0^\infty e^{-(\omega t + \theta\ell)} \int_0^\ell \mathbf{P}(B_\ell(t) \in (y, \ell]) dy d\ell dt, \quad (15)$$

which corresponds to evaluating $\tilde{S}(\theta, \omega, 0)$. We apply l'Hôpital's rule to (13) to obtain

$$\begin{aligned} \lim_{\psi \rightarrow 0} \tilde{S}(\theta, \omega, \psi) &= \lim_{\psi \rightarrow 0} \frac{1}{\theta\omega\psi} \frac{\tilde{g}(\omega)(\tilde{f}(\theta) - \tilde{f}(\theta + \psi))}{1 - \tilde{g}(\omega)\tilde{f}(\theta + \psi)} \\ &= -\frac{\tilde{g}(\omega)}{\theta\omega(1 - \tilde{g}(\omega)\tilde{f}(\theta))} \frac{d}{d\theta} \tilde{f}(\theta). \end{aligned} \quad (16)$$

It should be noted that the derivative of $\mathbf{E}[B_\ell(t)]$ may not exist. Indeed, if F is lattice then $\mathbf{E}[B_\ell(t)]$ will be a step function. In the case where the derivative does exist, we obtain its Laplace transform by multiplying (16) by θ to yield

$$\mathcal{L}_{\theta, \omega}\left\{\frac{\partial}{\partial\ell} \mathbf{E}[B_\ell(t)]\right\} = -\frac{\tilde{g}(\omega)}{\omega(1 - \tilde{g}(\omega)\tilde{f}(\theta))} \frac{d}{d\theta} \tilde{f}(\theta). \quad (17)$$

An example using a compound Poisson process

In this section we use Equation (17) to calculate the optimal limit for a transacting credit card customer who makes purchases according to a compound Poisson process with rate λ and purchase sizes that are exponentially distributed with parameter μ . The Laplace transforms of the inter-arrival time distribution and the purchase size distribution are

$$\tilde{f}(\theta) = \frac{\mu}{\mu + \theta}, \quad \text{and} \quad \tilde{g}(\omega) = \frac{\lambda}{\lambda + \omega} \quad (18)$$

so now equation (17) becomes

$$\mathcal{L}_{\theta, \omega}\left\{\frac{\partial}{\partial\ell} \mathbf{E}[B_\ell(t)]\right\} = \frac{\lambda\mu}{\omega(\theta + \mu)(\mu\omega + \theta(\lambda + \omega))}, \quad (19)$$

which can be inverted analytically to yield

$$\mathcal{L}_\theta \left\{ \frac{\partial}{\partial \ell} \mathbf{E}[B_\ell(t)] \right\} = \frac{1}{\theta} \left(\frac{\mu}{\mu + \theta} \right) \left(1 - \exp \left\{ \lambda t \left(\frac{\mu}{\mu + \theta} - 1 \right) \right\} \right). \quad (20)$$

It does not appear to be easy to perform further analytical inversion with respect to θ since the inverse transform of the exponential of a rational function is not a standard transform. As such, we resort to numerical inversion using the EULER algorithm as detailed in Abate and Whitt (1995).

We calculated the optimal limit using an interchange rate $\gamma = 0.0054$, a cost of funds $\nu = 0.0007$ and a statement period length $T = 30$. We took $\Lambda = (0, 5000]$ and used a bisection search to solve Equation (5). Table 1 shows the optimal limits calculated for a range of values of λ and μ .

An important factor in the assignment of credit limits is the customer's experience of having a purchase declined due to insufficient funds which, in our model, will result in the customer being barred from making further purchases until the end of the statement period. The probability of a purchase being declined is given by the tail function of $A(T)$ which, in the case of a compound Poisson process with exponential marks, has Laplace transform

$$\tilde{A}(\psi) = \frac{1}{\psi} \left(1 - \exp \left\{ \lambda T \left(\frac{\mu}{\mu + \psi} - 1 \right) \right\} \right), \quad \text{Re}(\psi) > -\mu. \quad (21)$$

The results of inverting Equation (21) at the optimal limits calculated in Table 1 are presented in Table 2. They show that the probability of a declined purchase remains constant when the rate parameter of the purchase size distribution changes. Indeed, by scaling the purchase size distribution by some $\alpha \in \mathbb{R}_+$, we scale the input process $A(t)$ and, as evidenced by Table 1, the optimal limit. The following proposition formalises this result.

Proposition 1 (Scaling property of the optimal limit). *Let $A'(t) \stackrel{d}{=} \alpha A(t)$ and let*

$$B_\ell(t) = \sup_{0 \leq u \leq t} \{A(u) : A(u) \leq \ell\} \quad \text{and} \quad B'_\ell(t) = \sup_{0 \leq u \leq t} \{A'(u) : A'(u) \leq \ell\}.$$

Then the solution to the optimisation problem

$$\hat{\ell}' = \arg \max_{\ell \in \Lambda} (\gamma \mathbf{E}[B'_\ell(t)] - \nu \ell) = \alpha \arg \max_{\ell \in \Lambda} (\gamma \mathbf{E}[B_\ell(t)] - \nu \ell) = \alpha \hat{\ell},$$

as long as $\alpha \hat{\ell} \in \Lambda$. Furthermore, we have that

$$\mathbf{P}(A'(t) > \hat{\ell}') = \mathbf{P}(A(t) > \hat{\ell}).$$

The proof of the above proposition is given in the appendix.

	1/ μ				
	20	40	60	80	100
λ	1	798.22	1 596.44	2 394.65	3 192.87
	2	1 470.40	2 940.80	4 411.21	5 881.61
	3	2 125.86	4 251.72	6 377.57	8 503.43
	4	2 772.63	5 545.26	8 317.90	11 090.53
	5	3 413.85	6 827.70	10 241.55	13 655.41
					17 069.26

Table 1: Table of values for the optimal limit. The values were calculated using a statement period length of $T = 30$, with $\gamma = 0.0054$ and $\nu = 0.0007$.

	1/ μ				
	20	40	60	80	100
λ	1	0.105 916 58	0.105 916 58	0.105 916 58	0.105 916 58
	2	0.112 186 71	0.112 186 71	0.112 186 71	0.112 186 71
	3	0.115 131 27	0.115 131 27	0.115 131 27	0.115 131 27
	4	0.116 938 21	0.116 938 21	0.116 938 21	0.116 938 21
	5	0.118 194 13	0.118 194 13	0.118 194 13	0.118 194 13

Table 2: Probability of the credit card customer experiencing a declined purchase when assigned the optimal limit.

Comparison with the newsvendor model

As mentioned in the introduction, the model we have formulated is similar to the single period newsvendor model with random demand. The objective of the newsvendor problem is to determine the number of newspapers to stock which will maximise the expected profit. Following the formulation in Porteus (2002), let $A(T)$ denote the random demand for newspapers in a single period of length T , ℓ the stock level of newspapers and γ and ν the unit profit and cost respectively. The expected revenue earned by the newsvendor in a period is $\gamma \mathbf{E}[A(T) \wedge \ell]$, representing the cases where the newsvendor has either ordered sufficiently-many newspapers to meet the demand $A(T)$, or an insufficient number, in which case he sells the entire stock ℓ ordered at the beginning of the period. The cost incurred by the newsvendor is simply the unit cost of each newspaper, ν , multiplied by the number of newspapers ℓ which he chooses to order. Thus, the problem is to determine

$$\ell^* := \arg \max_{\ell \in \mathbb{R}^+} \gamma \mathbf{E}[A(T) \wedge \ell] - \nu \ell, \quad (22)$$

which is similar to the problem formulated in Equation (4). The newsvendor problem has a well-known solution in terms of the distribution of $A(T)$. For comparison, we can rewrite (4) as

$$\hat{\ell} = \arg \max_{\ell \in \Lambda} \gamma \mathbf{E}[A(T) \wedge \ell - U] - \nu \ell, \quad (23)$$

where U is a random variable describing the undershoot of the process, conditional on the event that $A(T) > \ell$. Since an undershoot only occurs when the input process $A(t)$ exceeds ℓ before the end of the period T , we have

$$\begin{aligned}\mathbf{E}[A(T) \wedge \ell - U] &= \mathbf{P}(A(T) \leq \ell) \mathbf{E}[A(T) | A(T) \leq \ell] \\ &\quad + \mathbf{P}(A(T) > \ell) \mathbf{E}[\ell - U | A(T) > \ell].\end{aligned}\quad (24)$$

However, the solution using this formulation requires explicit knowledge of the distribution of U (or equivalently, $\ell - U$) which is, in general, difficult to obtain.

The difference between the two models lies in the fact that, in the case where a customer exceeds his or her credit limit, the balance of purchases made is (with probability one) strictly less than the limit ℓ , whereas the newsvendor always sells ℓ papers whenever the demand exceeds ℓ . The latter model would apply to the credit limit case if, whenever a customer attempts a purchase of value z that will take the current outstanding balance x over the credit limit ℓ , only $\ell - x$ is charged to the card and then no further purchases are allowed. This is clearly an unsatisfactory rule to use in our model since this would involve merchants only accepting partial payment for whatever goods were being purchased.

An improved model for credit card use would allow for the customer to continue to attempt purchases after a purchase has been declined since, in reality, a bank's card management system will permit any number of purchases to be made so long as the total value does not exceed ℓ . Let $\bar{B}_\ell(t)$ denote the value of the outstanding balance under this control policy. We can derive an integral equation for the tail function of $\bar{B}_\ell(t)$ by adding another case to Equation (8) to include the possibility that the process restarts from its original position if the first jump takes the process over ℓ . However, a closed-form expression for the Laplace transform of the tail function of $\bar{B}_\ell(t)$ is not immediately forthcoming when we include this third term.

Alternatively, we can obtain bounds on the limit that would be set when further purchases are allowed following rejection by using the newsvendor model and the model we have developed which prevents further purchases after the first declined purchase. We claim

$$\mathbf{E}[B_\ell(T)] \leq \mathbf{E}[\bar{B}_\ell(T)] \leq \mathbf{E}[A(T) \wedge \ell]. \quad (25)$$

That $\mathbf{E}[B_\ell(T)] \leq \mathbf{E}[A(T) \wedge \ell]$ follows from a direct comparison of the expectations in their integral form,

$$\int_0^\ell z F_A(dz) + (1 - F_A(\ell)) \int_0^\ell z F_Z(dz) \leq \int_0^\ell z F_A(dz) + (1 - F_A(\ell))\ell. \quad (26)$$

We reason that $\mathbf{E}[B_\ell(T)] \leq \mathbf{E}[\bar{B}_\ell(T)]$ since a balance control policy that allows for further purchases following a rejected purchase cannot decrease the expected balance. Similarly, $\mathbf{E}[\bar{B}_\ell(T)] \leq \mathbf{E}[A(T) \wedge \ell]$ since under the newsvendor control policy, a rejected purchase will always result in an outstanding balance of ℓ , but this is not necessarily so under the policy allowing for purchase retrials.

We now claim

$$\hat{\ell} \geq \bar{\ell} \geq \ell^* \quad (27)$$

where

$$\bar{\ell} := \arg \max_{\ell \in \Lambda} \{ \gamma \mathbf{E}[\bar{B}_\ell(T)] - \nu \ell \} \quad (28)$$

and

$$\ell^* := \arg \max_{\ell \in \Lambda} \{ \gamma \mathbf{E}[A(T) \wedge \ell] - \nu \ell \}, \quad (29)$$

which follows directly from (25).

The optimal limit in the newsvendor model is given by

$$\ell^* = \inf \left\{ \ell : F_A(\ell) \geq \frac{\gamma - \nu}{\gamma} \right\}, \quad (30)$$

and Table 3 shows the optimal limits calculated when the input process $A(t)$ is a compound Poisson process with arrival rate λ and exponential jumps with parameter μ . The differences between the limits obtained using the newsvendor model and the process $B_\ell(t)$ are shown in Table 4. These differences give a measure of the model error due to using the balance control policy where further purchases are prevented following an attempt to exceed the credit card limit.

An example using credit card transaction data

In this section, we apply the model we have developed to actual data from a credit card customer. Two datasets of anonymised credit card transactions were made available to the authors for the purposes of this research. The first dataset holds posted transactions, which are the approved purchases and payments, and also includes interest charges, fees, reversals and other automated transactions. The second dataset describes authorisations, which are the purchases and payments attempted by customers.

The posted transactions dataset describes the value and processing dates of 771,457 transactions made between 8 February 2011 and 27 February 2013 by 3,734 customers holding 3,971 accounts. Of the 771,457 transactions,

	$1/\mu$				
	20	40	60	80	100
λ	1	776.78	1,553.55	2,330.33	3,107.10
	2	1,449.38	2,898.76	4,348.14	5,797.52
	3	2,105.02	4,210.04	6,315.06	8,420.09
	4	2,751.91	5,503.81	8,255.72	11,007.63
	5	3,393.20	6,786.41	10,179.61	13,572.81
					16,966.02

Table 3: Table of values for ℓ^* , the optimal limit found using the newsvendor model, with $\gamma = 0.0054$, $\nu = 0.0007$ and $T = 30$.

	$1/\mu$				
	20	40	60	80	100
λ	1	21.44	42.89	64.33	85.77
	2	21.02	42.05	63.07	84.09
	3	20.84	41.67	62.51	83.34
	4	20.72	41.45	62.17	82.90
	5	20.65	41.30	61.95	82.59
					103.24

Table 4: Table of values for $\hat{\ell} - \ell^*$, the difference between the optimal limits calculated using the process $B_\ell(t)$ and the newsvendor model.

511,969 are retail purchase transactions and 84,503 are payments. In addition to the above, the dataset also contains identifying merchant information which allows us to categorise transactions by store type.

The authorisations dataset describes the value and transaction times, accurate to the second, of 405,844 transactions made between 7 February 2011 and 27 February 2013. The dataset also contains account credit limits and describes whether or not the transaction was approved or declined and, in the case of a decline, a code describing the reason (e.g. insufficient funds or an incorrectly entered PIN). These transactions were made by the same 3,734 customers across 4,333 credit card accounts. Due to issues encountered during the data extraction process, we were only able to match the authorisation transaction records with the posted transaction records of 2,246 customers. These customers attempted 288,423 purchases, of which 223,804 were approved.

For the purposes of illustrating the model developed in this paper, we extracted the transactions of a single customer who was identified as a transaction through the absence of interest charges to their account over the period. We filtered the transactions to include only those made at supermarkets since they account for a large proportion of purchases made on the credit card (306 out of 732) and are easily identified in both the authorisations and posted transactions dataset. The time series was modified to exclude transactions that were declined due to a POS device error or an incorrect PIN entry. A preliminary analysis of the supermarket transactions of several

card-holders revealed occasional clustering of transactions in time. This could be explained by a number of customer behaviours. For example, a customer may visit a supermarket only to find that some of the items they intended on purchasing are not available, so they buy the items that are in stock and then visit another supermarket nearby to purchase the remaining items. It could also be due to a customer forgetting some items, and quickly returning to the same store to purchase them. With this in mind, transactions made within an hour of each other were combined into a single transaction with the total value of those transactions.

The customer made 306 purchases at various supermarkets over a period of 473 days which totalled \$11,469.44. This equates to approximately \$37.36 per transaction or \$24.25 per day. In a 30-day period, this totals \$727.50 in purchases, which is far less than the account credit limit of \$5,000.

We fitted a Γ -distribution to the purchase values of the modified time series and estimated the shape and scale parameters using maximum likelihood estimation. Using the two-sided Kolmogorov-Smirnov test statistic

$$D_n = \sup_x |F_n(x) - F(x)| \quad (31)$$

where $F_n(x)$ is the empirical distribution function and $F(x)$ is the distribution function of the fitted Γ -distribution, we found the fit to be statistically significant at the 0.05 level as evidenced by the result in Table 5. Finding an appropriate distribution for the inter-transaction times was not so straightforward, so for the purposes of this example, we assume the inter-transaction times follow an exponential distribution with parameter λ , which we estimated from the reciprocal of the mean of the inter-purchase times to be $\hat{\lambda} = 0.6451 \pm 0.0369$. We further assumed independence of the purchase values and the inter-purchase times, but note that this assumption could be tested by computing the coherence between the inter-purchase times and the purchase values (see theorem 4.4 in Brillinger (2012)). Some degree of dependence between inter-purchase time and purchase value is likely, particularly with supermarket transactions, since a large inter-purchase time would indicate that a customer has not visited a supermarket for a while, and hence the next purchase is likely to be a large one. Substituting

Statistic	Estimate
D_n	0.0350
p -value	0.8623
$\hat{\mu}$ (shape)	2.8946 ± 0.2258
\hat{k} (scale)	0.0769 ± 0.0065

Table 5: Kolmogorov-Smirnov test statistics and Γ -distribution shape and scale parameter estimates for the purchase value distribution.

$$\tilde{g}(\omega; \lambda) = \left(\frac{\lambda}{\lambda + \omega} \right) \quad \text{and} \quad \tilde{f}(\theta; \mu, k) = \left(\frac{\mu}{\mu + \theta} \right)^k \quad (32)$$

into Equation (17) and inverting once from ω to t , we have for the Laplace transform of the expectation and its derivative

$$\mathcal{L}_\theta \{ \mathbf{E}[B_\ell(t)] \} = \frac{k}{\theta(\mu + \theta)} \left(\frac{\mu}{\mu + \theta} \right)^k \frac{1 - e^{\lambda t \left(\left(\frac{\mu}{\mu + \theta} \right)^k - 1 \right)}}{1 - \left(\frac{\mu}{\mu + \theta} \right)^k} \quad (33)$$

and

$$\mathcal{L}_\theta \left\{ \frac{\partial}{\partial \ell} \mathbf{E}[B_\ell(t)] \right\} = \frac{k}{\mu + \theta} \left(\frac{\mu}{\mu + \theta} \right)^k \frac{1 - e^{\lambda t \left(\left(\frac{\mu}{\mu + \theta} \right)^k - 1 \right)}}{1 - \left(\frac{\mu}{\mu + \theta} \right)^k}. \quad (34)$$

For calculations using the newsvendor model, we use the Laplace transform of the tail function of the compound Poisson process with Γ -distributed jumps,

$$\tilde{A}_\Gamma(\psi) = \frac{1}{\psi} \left(1 - \exp \left\{ \lambda T \left(\left(\frac{\mu}{\mu + \psi} \right)^k - 1 \right) \right\} \right), \quad \text{Re}(\psi) > -\mu \quad (35)$$

and

$$\mathcal{L}_\theta \{ \mathbf{E}[A(T) \wedge \ell] \} = \int_0^\infty e^{-\theta y} \int_0^\ell \mathbf{P}(A(T) > y) dy d\ell = \frac{1}{\theta} \tilde{A}_\Gamma(\theta). \quad (36)$$

Again, we assume an interchange rate $\gamma = 0.0054$, cost of funds $\nu = 0.0007$ and statement period length $T = 30$. Substituting the estimated parameters $\hat{\lambda}$, $\hat{\mu}$ and \hat{k} into Equations (34)–(35), we obtain the results in Table 6. We again used a bisection search and the EULER algorithm to calculate the optimal limits. The table shows the expected balance, expected profit and probability of a declined purchase at the original limit, the upper and lower bounds of the optimal limit and a revised limit. The bounds on the optimal limit accord with the average monthly supermarket spend of the customer. Recall that we stated that the customer spent \$727.50 in a 30-day period; the upper and lower limits yield an expected balance of just over \$714. The increase in profitability is substantial, but we note that this is somewhat artificial given we have restricted our analysis to only those purchases made at supermarkets. The revised limit is proposed since most card-issuers offer limits in multiples of \$500. As shown in the table, the deviation from profit at optimality is negligible, but there is slightly smaller chance of the customer experiencing a declined purchase. Although the results in Table 6 show a marked increase in profitability as a result of lowering the credit limit to the optimal limit, it should be noted that doing so would substantially

	Original	Optimal	Revised
Limit	\$5,000.00	[\$947.83, \$973.81]	\$1,000.00
Expected balance	\$728.64	[\$714.06, \$714.09]	[\$717.13, \$719.64]
Expected profit	\$0.44	[\$3.17, \$3.19]	[\$3.17, \$3.19]
Probability of decline	0.0000	[0.1060, 0.1296]	0.0857

Table 6: Expected balance, expected profit and probability of a declined purchase at the original limit, upper and lower bounds of the optimal limit and a proposed revised limit.

increase the probability that the customer will experience a declined purchase if their purchasing behaviour remains unchanged. This is undoubtedly a poor experience for the customer and the consequences of this for both the customer and the bank should be considered before any change to the customer’s credit limit is made.

Discussion

The model we have presented makes a number of simplifying assumptions. As mentioned in the introduction, the assumption of transacting behaviour is both valid and useful since most credit card portfolios are primarily composed of customers exhibiting this behaviour and they form a significant source of revenue through interchange. We can extend the model to include the possibility of partial repayment of the outstanding balance by including another term in (1) which describes how partial repayment generates interest. To then derive the resulting optimal limit requires additional assumptions on payment behaviour and the value of new purchases when the account retains a partially unpaid balance. We regard this as a worthwhile avenue for future research given its applicability in credit management.

We assumed that the attempted purchase process was a marked point process with inter-purchase time distribution G and purchase size distribution F , and that these distributions remain unchanged with respect to the outstanding balance and the credit limit. A more realistic model would include state-dependence, as our data indicates that customers either reduce their purchase frequency or purchase size as they near their credit limit. Another modification would be to split $A(t)$ into a series of marked point processes, each modelling different types of purchases such as retail, restaurants, supermarkets, or cash advances.

Related to the above, our calculation of the optimal limit assumes that $A(t)$ will remain unchanged in the event of a change in the credit limit: an improved model may factor in the reward associated with an increased limit (customers may increase their overall attempted spend) or, correspondingly, a penalty associated with a decrease (a customer may decide to cancel their

card). It was demonstrated in Soman and Cheema (2002) that increasing credit limits resulted in increased customer spend only in some portfolio segments. However, there is relatively little published research on the effect of credit limit decreases on customer purchasing behaviour. Also, implementing the resulting optimal limit exactly may prove difficult as most banks only offer limits to customers that are multiples of \$500, for both customer experience and systems reasons. Nonetheless, we regard the model as a useful complement to existing limit setting strategies for understanding the effect of limit changes on profitability and customer experience.

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Proof of the scaling property of $\hat{\ell}$

By the assumption that $A'(t) \stackrel{d}{=} \alpha A(t)$,

$$\begin{aligned}\hat{\ell}' &= \arg \max_{\ell'} \left\{ \gamma \mathbf{E} [B'_{\ell'}(t)] - \nu \ell' \right\} \\ &= \arg \max_{\ell'} \left\{ \gamma \mathbf{E} \left[\sup_{0 \leq u \leq t} \{A'(u) : A'(u) \leq \ell'\} \right] - \nu \ell' \right\} \\ &= \alpha \arg \max_{\ell'} \left\{ \gamma \mathbf{E} \left[\sup_{0 \leq u \leq t} \left\{ A(u) : A(u) \leq \frac{\ell'}{\alpha} \right\} \right] - \nu \frac{\ell'}{\alpha}, \right\}.\end{aligned}$$

Making the substitution $\ell = \ell'/\alpha$,

$$\begin{aligned}\hat{\ell}' &= \alpha \arg \max_{\ell} \left\{ \gamma \mathbf{E} \left[\sup_{0 \leq u \leq t} \{A(u) : A(u) \leq \ell\} \right] - \nu \ell \right\} \\ &= \alpha \arg \max_{\ell} \left\{ \gamma \mathbf{E}[B_\ell(t)] - \nu \ell \right\} = \alpha \hat{\ell},\end{aligned}$$

which shows that the optimal limit scales with α . Since $\hat{\ell}' = \alpha \hat{\ell}$, we have

$$\mathbf{P}(A'(t) > \hat{\ell}') = \mathbf{P}(\alpha A(t) > \alpha \hat{\ell}) = \mathbf{P}(A(t) > \hat{\ell}),$$

which shows that the blocking probabilities remain the same.